Flexible and Rigid Labelings of Graphs

Jan Legerský

RISC JKU Linz, Austria

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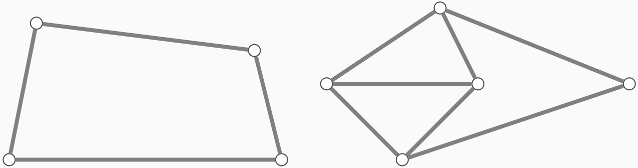

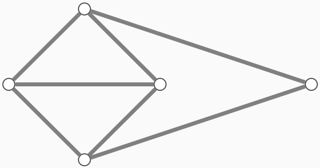

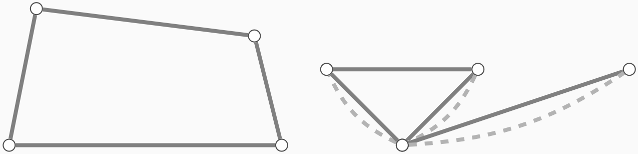

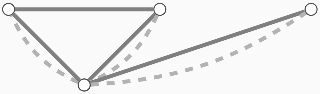

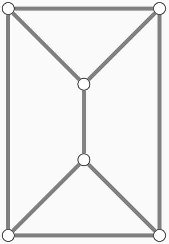

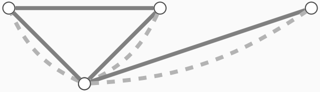

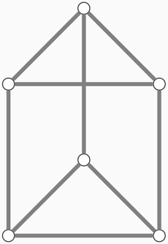


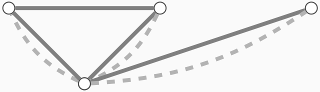
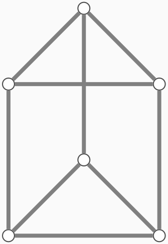
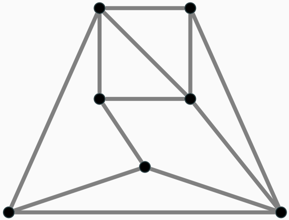

Outline

- Existence of Flexible Labelings
- Movable Graphs
- On the Classification of Motions
- Number of Real Realizations compatible with a Rigid Labeling

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  - ▷ E. Bartzos, I. Z. Emiris, J. Legerský, and E. Tsigaridas. On the Maximal Number of Real Embeddings of Spatial Minimally Rigid Graphs. *ISSAC 2018*.
  - ▷ E. Bartzos, I. Z. Emiris, J. Legerský, and E. Tsigaridas. On the maximal number of real embeddings of minimally rigid graphs in $\mathbb{R}^2$, $\mathbb{R}^3$ and S^2, *arXiv*, 2018.

Flexible and rigid labelings

Let $\lambda : E_G \to \mathbb{R}_+$ be an edge labeling of a graph $G = (V_G, E_G)$.
A *realization* $\rho : V_G \to \mathbb{R}^2$ is compatible with λ if
$\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges uv in E_G.

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The labeling λ is called

- *flexible* if there are infinitely many non-congruent compatible realizations, or
- *rigid* if the number of non-congruent compatible realizations is positive and finite.

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- *proper flexible* if there are infinitely many non-congruent injective compatible realizations, or
- *rigid* if the number of non-congruent compatible realizations is positive and finite.

A graph is called *movable* if it has a proper flexible labeling.

Algebraic formulation

$$(x_{\bar{u}}, y_{\bar{u}}) = (0, 0)$$
$$(x_{\bar{v}}, y_{\bar{v}}) = (\lambda(\bar{u}\bar{v}), 0)$$
$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2, \quad \forall\, uv \in E_G$$

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$\implies$ λ is proper flexible,

A 1-dimensional irreducible subset of the zero set is called an *algebraic motion*.

Laman graphs

Definition

A graph G is called *Laman* if $|E_G| = 2|V_G| - 3$, and $|E_H| \leq 2|V_H| - 3$ for every subgraph H of G.

Theorem (Pollaczek-Geiringer, Laman)

A labeling of a graph G induced by a generic realization of G is rigid if and only if G is spanned by a Laman graph.

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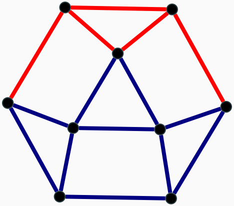
Are there any Laman graphs with a (proper) flexible labeling?

Existence of Flexible Labelings

NAC-colorings

Definition

A coloring of edges $\delta : E_G \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.



Combinatorial characterization

Theorem (GLS)

A connected graph with at least one edge has a flexible labeling if and only if it has a NAC-coloring.

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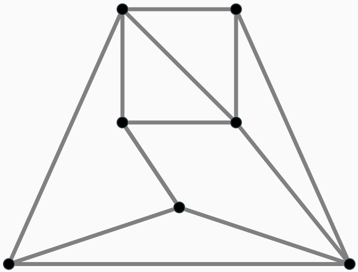

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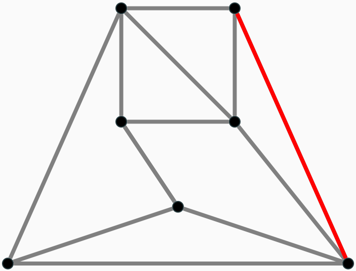

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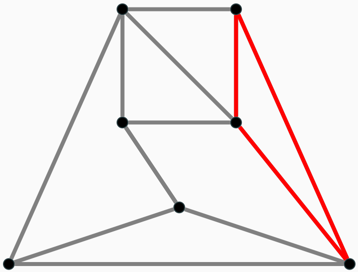

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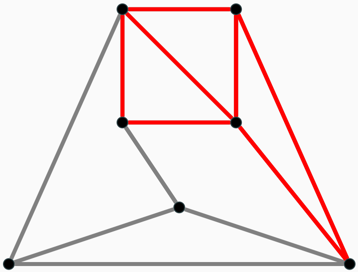

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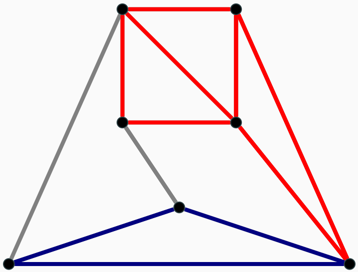

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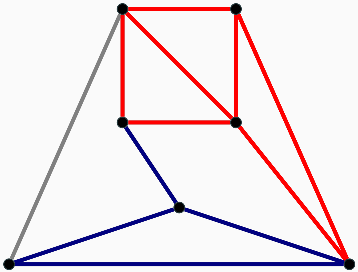

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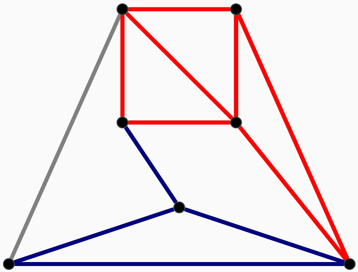
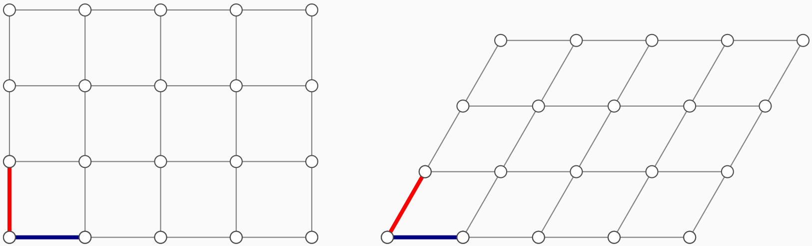
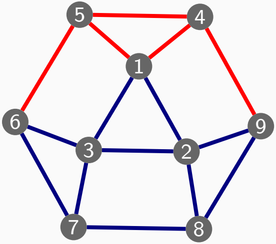


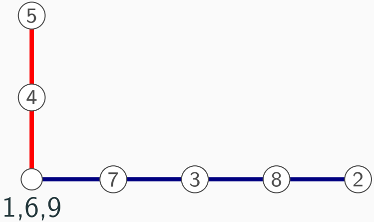
$\Longrightarrow$ no flexible labeling

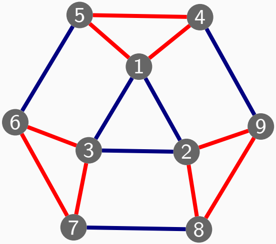
Grid construction

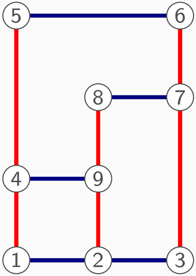


Example

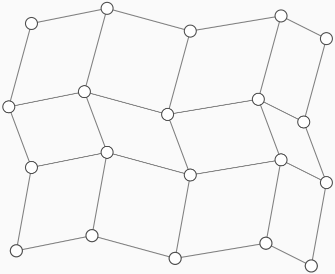


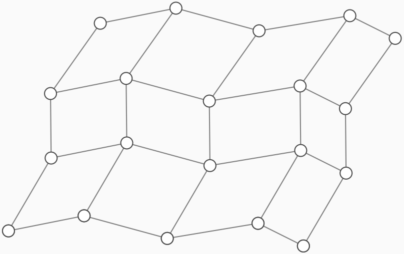






Grid construction II





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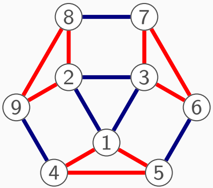

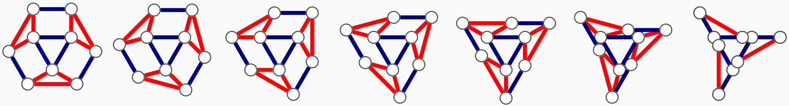

Functions $W_{u,v}$ and $Z_{u,v}$

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For every cycle $(u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{i=1}^{n} W_{u_i,u_{i+1}} = 0 \quad \text{and} \quad \sum_{i=1}^{n} Z_{u_i,u_{i+1}} = 0$$

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Lemma (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ). If $\alpha \in \mathbb{Q}$ and ν is a valuation of $F(\mathcal{C})$ trivial on $\mathbb{C}$ such that there exist edges $\bar{u}\bar{v}, \hat{u}\hat{v}$ with $\nu(W_{\bar{u},\bar{v}}) = \alpha$ and $\nu(W_{\hat{u},\hat{v}}) > \alpha$, then $\delta : E_G \to \{red, blue\}$ given by

$$\delta(uv) = red \iff \nu(W_{u,v}) > \alpha\,,$$
$$\delta(uv) = blue \iff \nu(W_{u,v}) \le \alpha\,.$$

is a NAC-coloring, called active.

Movable Graphs

Constant distances

Lemma (GLS)

Let G be a graph and $u, v \in V_G$ be such that $uv \notin E_G$. If there exists a uv-path P in G such that P is unicolor for all NAC-colorings of G, then G is movable if and only if $G' = (V_G, E_G \cup \{uv\})$ is movable.

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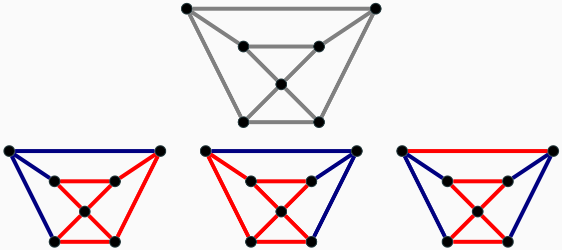

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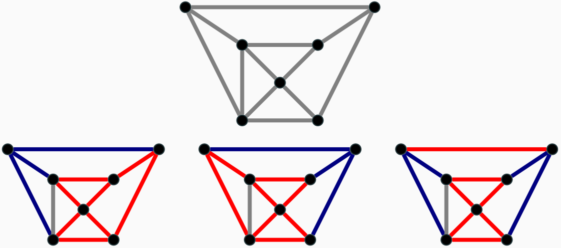

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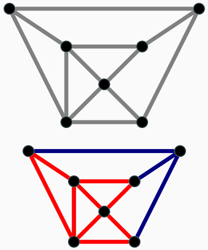
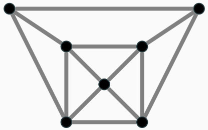


Constant distances

Lemma (GLS)

Let G be a graph and $u, v \in V_G$ be such that $uv \notin E_G$. If there exists a uv-path P in G such that P is unicolor for all NAC-colorings of G, then G is movable if and only if $G' = (V_G, E_G \cup \{uv\})$ is movable.

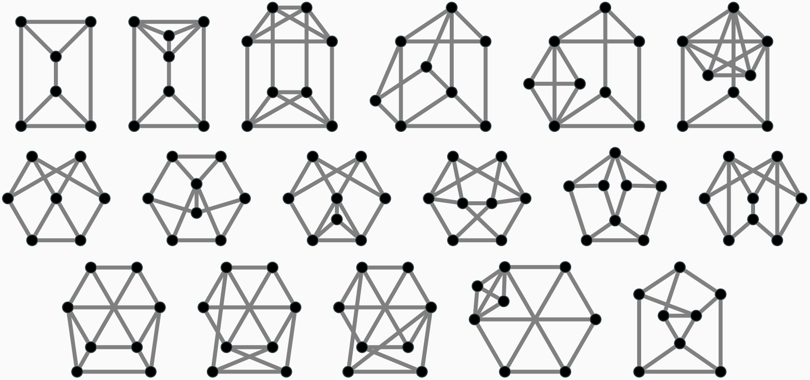


$\implies$ constant distance closure

Movable graphs up to 8 vertices

Theorem (GLS)

The maximal movable graphs with at most 8 vertices that are spanned by a Laman graph and have no vertex of degree two are the following: $K_{3,3}, K_{3,4}, K_{3,5}, K_{4,4}$ *or*



Embedding in $\mathbb{R}^3$

Lemma (GLS)

If there exists an injective realization of G in $\mathbb{R}^3$ *such that every edge is parallel to one of the four vectors* $(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)$, *then G is movable.*

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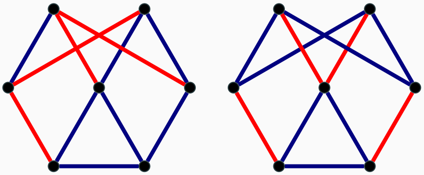
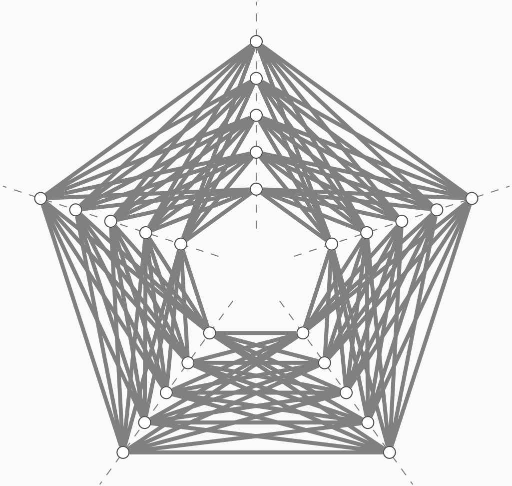
Moreover, there exists an algebraic motion of G with exactly two active NAC-colorings. Two edges are parallel in the embedding ω if and only if they receive the same pair of colors in the two active NAC-colorings.

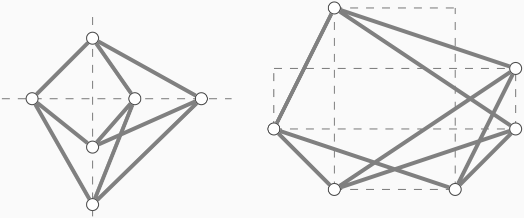


Counterexample



On the Classification of Motions

Classification of motions



Dixon (1899), Walter and Husty (2007)

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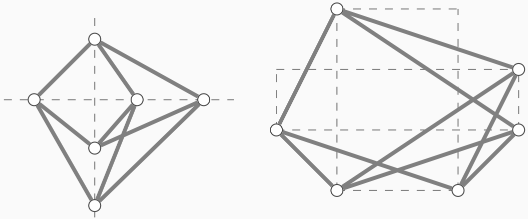

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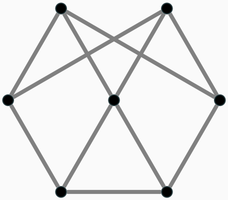

Q_1

Active NAC-colorings of quadrilaterals

Quadrilateral	Motion	Active NAC-colorings
Rhombus	parallel	
	degenerate	 resp. 
Parallelogram		
Antiparallelogram		
Deltoid	nondegenerate	
	degenerate	
General		

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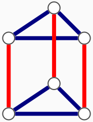

Leading coefficient system

Assume a valuation that gives only one active NAC-coloring
$\implies$ Laurent series parametrization.

For every cycle $C = (u_1, \ldots, u_n, u_{n+1} = u_1)$:

$$\sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{red}}} (\underbrace{w_{u_i u_{i+1}} t + \text{h.o.t.}}_{W_{u_i, u_{i+1}}}) + \sum_{\substack{i \in \{1,\ldots,n\} \\ \delta(u_i u_{i+1}) = \text{blue}}} (\underbrace{w_{u_i u_{i+1}} + \text{h.o.t.}}_{W_{u_i, u_{i+1}}}) = 0\,.$$

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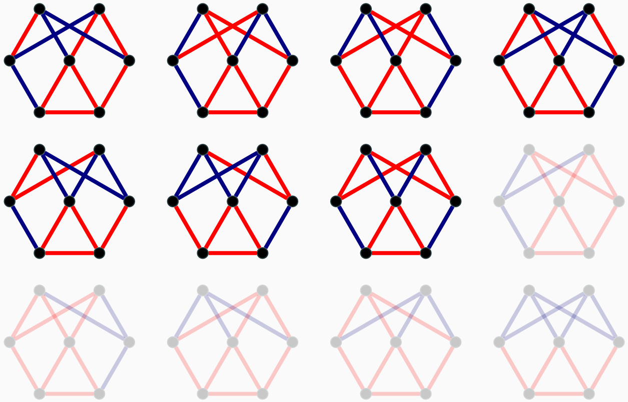
For all $uv \in E_G$:

$$w_{uv} z_{uv} = \lambda_{uv}^2\,.$$

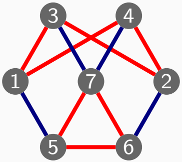
$\Longrightarrow$ elimination using Gröbner basis provides an equation in λ_{uv}'s.

Singleton NAC-colorings

If a valuation yields two active NAC-colorings δ, δ', then the set $\{(\delta(e), \delta'(e)) \colon e \in E_G\}$ has 3 elements.



Triangle in $Q1$



$$\implies \lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0\,,$$

$$r = \lambda_{24}^2 - \lambda_{23}^2,\; s = \lambda_{14}^2 - \lambda_{13}^2$$

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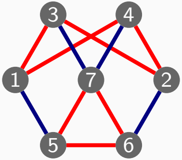

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Considering the equation as a polynomial in r, the discriminant is

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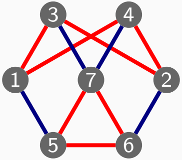

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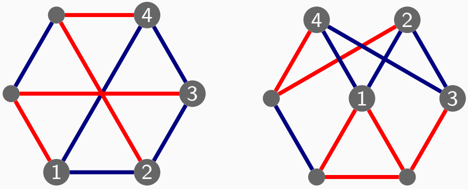
Theorem (GLS)

The vertices 5, 6 and 7 are collinear for every proper flexible labeling of Q_1.

Orthogonal diagonals



Lemma (GLS)

If there is an active NAC-coloring δ of an algebraic motion of (G, λ) such that a 4-cycle $(1, 2, 3, 4)$ is blue and there are red paths from 1 to 3 and from 2 to 4, then

$$\lambda_{12}^2 + \lambda_{34}^2 = \lambda_{23}^2 + \lambda_{14}^2 \,,$$

namely, the 4-cycle $(1, 2, 3, 4)$ has orthogonal diagonals.

Ramification formula

Theorem (GLS)

Let $\mathcal{C}$ be an algebraic motion of (G, λ) with the set of active NAC-colorings N. There exist $\mu_\delta \in \mathbb{Z}_{\geq 0}$ for all NAC-colorings δ of G such that:

1. *$\mu_\delta \neq 0$ if and only if $\delta \in N$, and*
2. *for every 4-cycle (V_i, E_i) of G, there exists a positive integer d_i such that*

$$\sum_{\substack{\delta \in \mathrm{NAC}_G \\ \delta|_{E_i} = \delta'}} \mu_\delta = d_i \qquad \text{for all } \delta' \in \{\delta|_{E_i} : \delta \in N\}.$$

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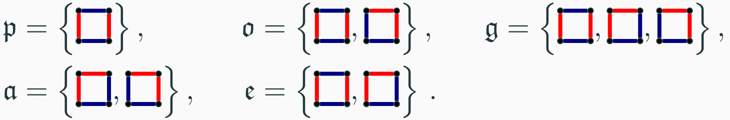
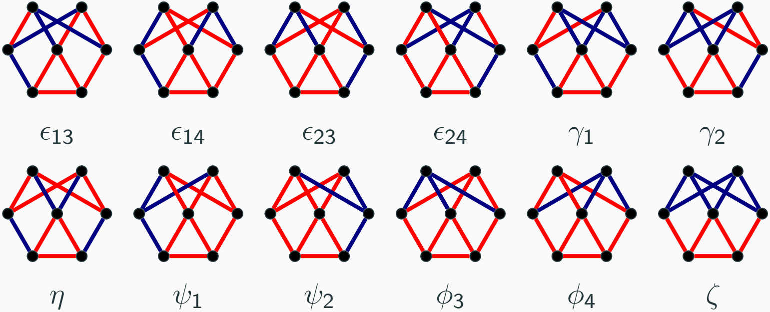
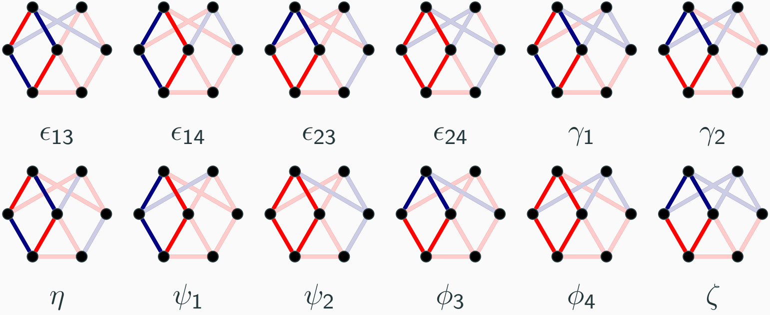
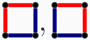


Example

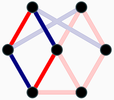

ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2
η ψ_1 ψ_2 ϕ_3 ϕ_4 ζ

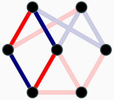
Example



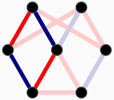
Antiparallelogram () $\Longrightarrow$

Example

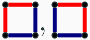




ϵ_{13}	ϵ_{14}	ϵ_{23}	ϵ_{24}	γ_1	γ_2

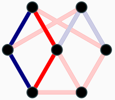


η	ψ_1	ψ_2	ϕ_3	ϕ_4	ζ

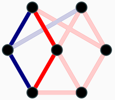
Antiparallelogram () $\implies$

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

Example



ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2



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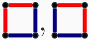
Antiparallelogram () $\Longrightarrow$

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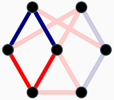
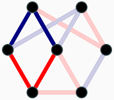
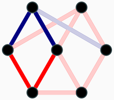
$$\mu_{\epsilon_{14}} + \mu_{\psi_1}$$

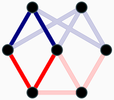
Example





ϵ_{13} ϵ_{14} ϵ_{23} ϵ_{24} γ_1 γ_2





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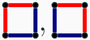
Antiparallelogram () $\Longrightarrow$

$$\mu_{\epsilon_{13}} = \mu_{\gamma_1} = \mu_\eta = 0$$

$$\mu_{\epsilon_{14}} + \mu_{\psi_1} = \mu_{\epsilon_{23}} + \mu_{\gamma_2} + \mu_{\phi_3} + \mu_\zeta$$

Classification of motions

- Find all possible types of motions of quadrilaterals with consistent μ_δ's

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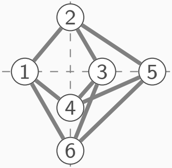
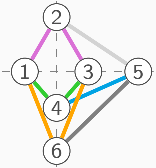
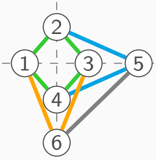
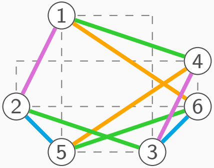
Implementation – SageMath package `FlexRiLoG` (`https://github.com/Legersky/flexrilog`)







4-cycles	active NAC-colorings	#	
$\mathfrak{ggggggggg}$	$\mathrm{NAC}_{K_{3,3}}$	1	
$\mathfrak{ooogggggg}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}, \epsilon_{16}, \epsilon_{36}, \omega_1, \omega_3\}$	6	Dixon I
$\mathfrak{pooggogge}$	$\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}, \epsilon_{14}\}$	9	
$\mathfrak{pgggaggag}$	$\{\epsilon_{12}, \epsilon_{34}, \omega_5, \omega_6\}$	18	Dixon II



Classification of motions of Q_1

4-cycles	active NAC-colorings	#	type	dim.
$\mathfrak{pggpgpg}$	$\{\epsilon_{13}, \epsilon_{24}, \eta\}$	2	I	4
$\mathfrak{poapope}$	$\{\epsilon_{13}, \eta\}$	4	$\subset$ I, IV$_-$, V, VI	2
$\mathfrak{peepapa}$	$\{\epsilon_{13}, \epsilon_{24}\}$	2	$\subset$ I, II, III	2
$\mathfrak{ogggggg}$	$\{\epsilon_{ij}, \gamma_1, \gamma_2, \psi_1, \psi_2\}$	1	II$_-$ $\cup$ II$_+$	5
$\mathfrak{peegggg}$	$\{\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}\}$	1	$\subset$ II$_-$, II$_+$	4
$\mathfrak{oggpgga}$	$\{\epsilon_{13}, \epsilon_{24}, \gamma_1, \psi_2\}$	4	$\subset$ II$_-$	3
$\mathfrak{oggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \gamma_1, \gamma_2\}$	2	$\subset$ II$_-$, deg.	2
$\mathfrak{ogggaga}$	$\{\epsilon_{13}, \epsilon_{24}, \psi_1, \psi_2, \zeta\}$	2	III	3
$\mathfrak{ggapggg}$	$\{\epsilon_{13}, \eta, \phi_4, \psi_2\}$	4	IV$_-$ $\cup$ IV$_+$	4
$\mathfrak{ggaegpe}$	$\{\epsilon_{13}, \eta, \gamma_2, \phi_3\}$	4	V	3
$\mathfrak{pggegge}$	$\{\epsilon_{13}, \epsilon_{23}, \eta, \zeta\}$	2	VI	3

Number of Real Realizations compatible with a Rigid Labeling

Number of real realizations

How many realizations of a Laman graph are compatible with a given rigid labeling?

$$
\begin{aligned}
(x_{\bar{u}}, y_{\bar{u}}) &= (0, 0) \\
(x_{\bar{v}}, y_{\bar{v}}) &= (\lambda(\bar{u}\bar{v}), 0) \\
(x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda(uv)^2, \qquad \forall\, uv \in E_G
\end{aligned}
$$

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$\Longrightarrow$ Bounded by the number of the complex solutions.

Goal — specify edge lengths with many real solutions.

Laman graphs with many real realizations

- 6 vertices: Borcea and Streinu '04
- 7 vertices: Emiris and Moroz '11
- Capco, Gallet, Grasegger, Koutschan, Lubbes and Schicho '18

# vertices	6	7	8	9	10	11	12
minimum	16	32	64	128	256	512	1024
maximum ($\mathbb{C}$)	24	56	136	344	880	2288	6180
maximum ($\mathbb{R}$)	24	56					

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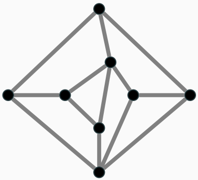
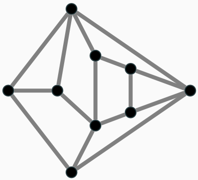
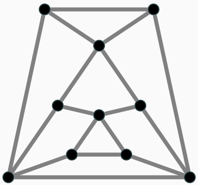
maximum ($\mathbb{R}$)	24	56	136	344	≥ 860	–	–



$G_{2,8}^{\max}$



$G_{2,9}^{\max}$



$G_{2,10}^{\max}$

3D – Geiringer graphs

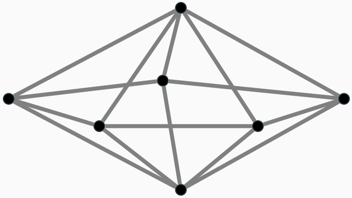
- 6 vertices: Emiris, Tsigaridas and Varvitsiotis '13
- Grasegger, Koutschan, Tsigaridas '18

# vertices	6	7	8	9	10
minimum ($\mathbb{C}$)	8	16	24	48	76
maximum ($\mathbb{C}$)	16	48	160	640	2560
maximum ($\mathbb{R}$)	16				

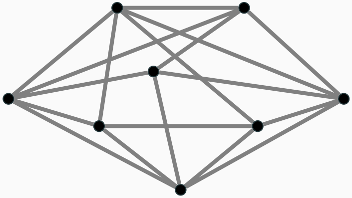
3D – Geiringer graphs

- 6 vertices: Emiris, Tsigaridas and Varvitsiotis '13
- Grasegger, Koutschan, Tsigaridas '18

# vertices	6	7	8	9	10
minimum ($\mathbb{C}$)	8	16	24	48	76
maximum ($\mathbb{C}$)	16	48	160	640	2560
maximum ($\mathbb{R}$)	16	48	≥ 132	–	–



$G_{3,7}^{\max}$

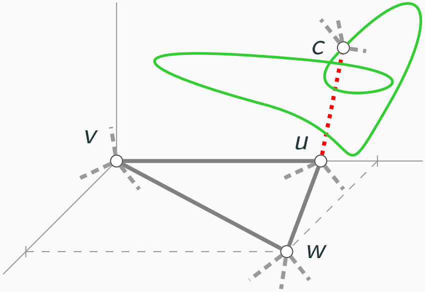


$G_{3,8}^{\max}$

Coupler Curves

Removing an edge *uc* yields a flexible structure.

The curve traced by the vertex *c* is called a *coupler curve*.



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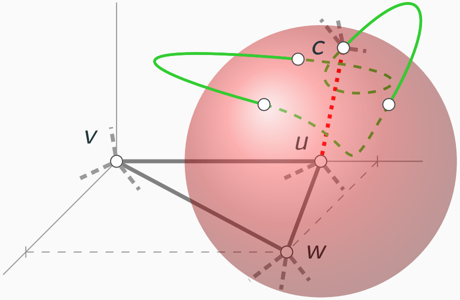
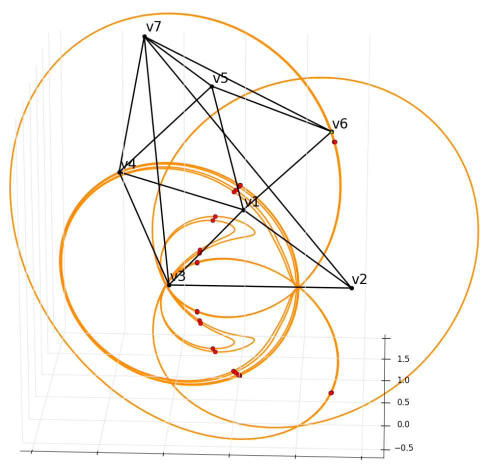
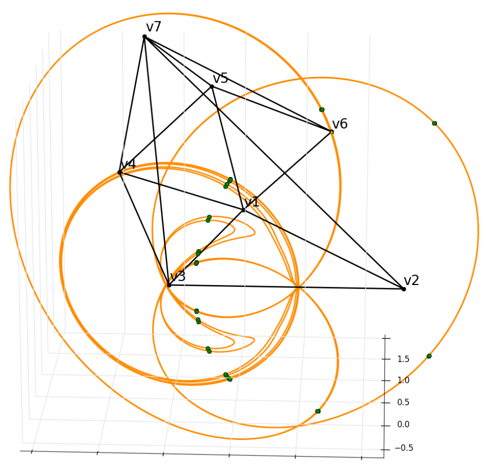


Example



Example



Summary

Future work and open questions

- Existence of Flexible Labelings
- Movable Graphs
- On the Classification of Motions
- Number of Real Realizations compatible with a Rigid Labeling

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  - ▷ NAC-colorings of Laman graphs
  - ▷ NAC-colorings yielding proper flexible labelings
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  - ▷ Larger graphs, better bounds
- Realizations on the Sphere
  - ▷ NAP-colorings, classification of motions of $K_{3,3}$

Thank you

jan.legersky@risc.jku.at

jan.legersky.cz