

# Paradoxical Flexibility of Reflection Symmetric Frameworks

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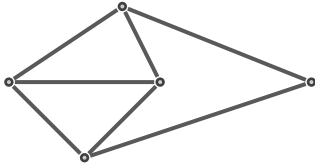
Jan Legerský

*Sean Dewar and Georg Grasegger*

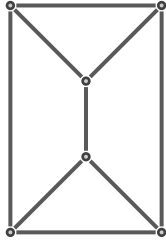
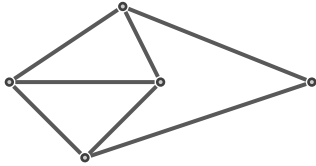
Czech Technical University in Prague, FIT, Czech Republic

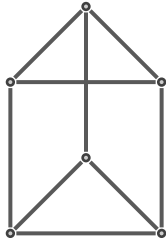
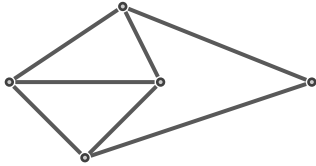
SIAM Conference on Applied Algebraic Geometry (AG25)

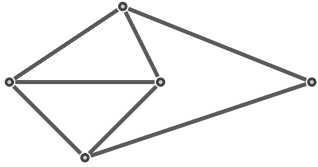
Madison, Wisconsin, USA, July 8, 2025











## Definition

A map  $p : V_G \rightarrow \mathbb{R}^2$  for a graph  $G = (V_G, E_G)$  such that  $p(u) \neq p(v)$  for every edge  $uv \in E_G$  is called a *quasi-injective realization*.

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A *flex* of the *framework*  $(G, p)$  is a continuous path  $t \mapsto p_t$ ,  $t \in [0, 1)$ , in the space of realizations of  $G$  such that  $p_0 = p$  and for all  $t \in [0, 1)$  and all edges  $uv \in E_G$

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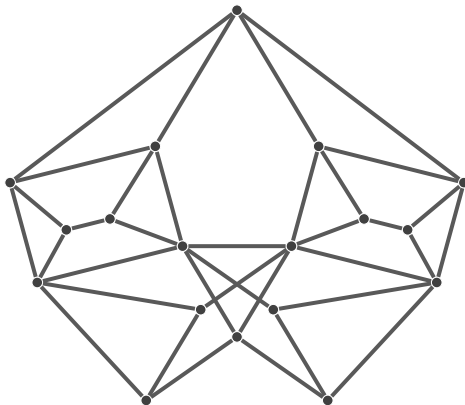
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The framework  $(G, p)$  is *flexible* if it has a non-trivial flex. Otherwise it is called *rigid*.



Does there exist a reflection-symmetric flexible realization?

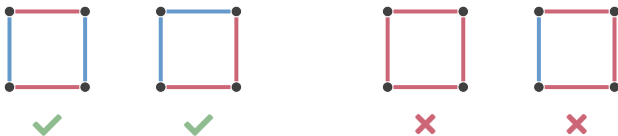
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A *NAC-coloring* of a graph  $G$  is a surjective edge coloring by red and blue such that for every cycle in  $G$ , either all edges have the same color, or there are at least two edges of each color.



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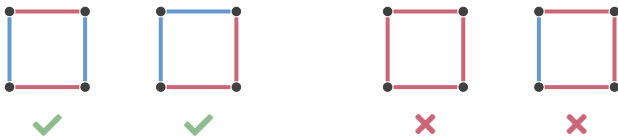


## Theorem (Grasegger, L., Schicho, 2019)

*A connected graph has a flexible realization in the plane if and only if it has a NAC-coloring.*

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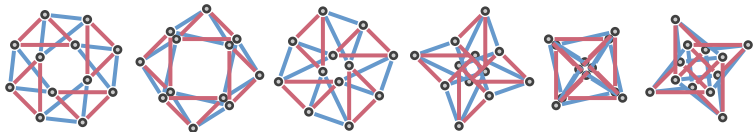
*A connected graph has a flexible realization in the plane if and only if it has a NAC-coloring.*

## Theorem (Garamvölgyi, 2022)

*The existence of a NAC-coloring is NP-complete.*

## Theorem (Dewar, Grasegger, L., 2022)

*A connected  $n$ -fold rotationally symmetric graph has an  $n$ -fold rotationally symmetric flexible realization if and only if the graph has a NAC-coloring that is invariant under the rotation and no two distinct blue, resp. red, partially invariant components are connected by an edge.*



# Reflection symmetry

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A quasi-injective realisation  $p : V_G \rightarrow \mathbb{R}^2$  of  $G$  is said to be *reflection-symmetric* if  $p(\sigma v) = \tau p(v)$  for every vertex  $v \in V_G$ , where

$$\tau := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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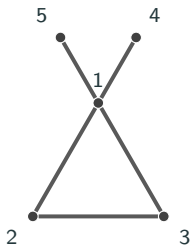
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A flex  $p_t$  of  $(G, p)$  is *reflection-symmetric* if for each  $t \in [0, 1]$ , the framework  $(G, p_t)$  is reflection-symmetric.

The framework is *reflection-symmetric flexible* if it has a non-trivial reflection-symmetric flex.



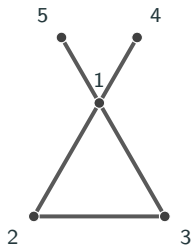
$$\sigma_1(1) = 1 \quad \sigma_2(1) = 1 \quad \sigma_3(1) = 1$$

$$\sigma_1(2) = 3 \quad \sigma_2(2) = 2 \quad \sigma_3(2) = 3$$

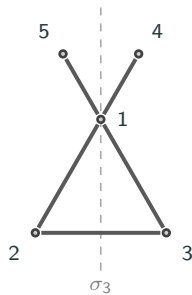
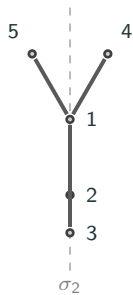
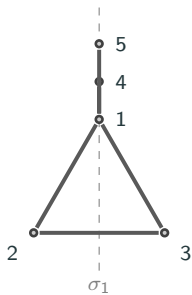
$$\sigma_1(3) = 2 \quad \sigma_2(3) = 3 \quad \sigma_3(3) = 2$$

$$\sigma_1(4) = 4 \quad \sigma_2(4) = 5 \quad \sigma_3(4) = 5$$

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$$\begin{array}{lll}
 \sigma_1(1) = 1 & \sigma_2(1) = 1 & \sigma_3(1) = 1 \\
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 \end{array}$$



## **(Pseudo)-RS-colorings**

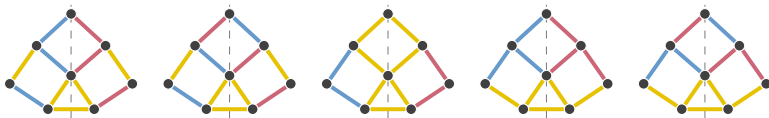
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## Definition

Let  $G$  be a graph with reflection  $\sigma$ . An edge coloring  $\delta : E_G \rightarrow \{\text{red, blue, gold}\}$  of  $G$  is a *pseudo-RS-coloring* if:

1.  $\{\text{red, blue}\} \subseteq \delta(E_G) \subseteq \{\text{red, blue, gold}\}$ ,
2. changing gold to blue results in a NAC-coloring,
3. changing gold to red results in a NAC-coloring,
4.  $\delta(e) = \text{red}$  if and only if  $\delta(\sigma e) = \text{blue}$  for all  $e \in E_G$ .

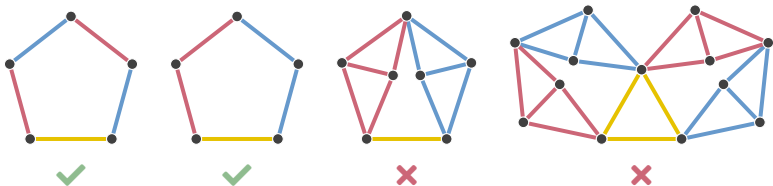
A cycle in  $G$  is *almost red-blue* if it has exactly one gold edge.



## Definition

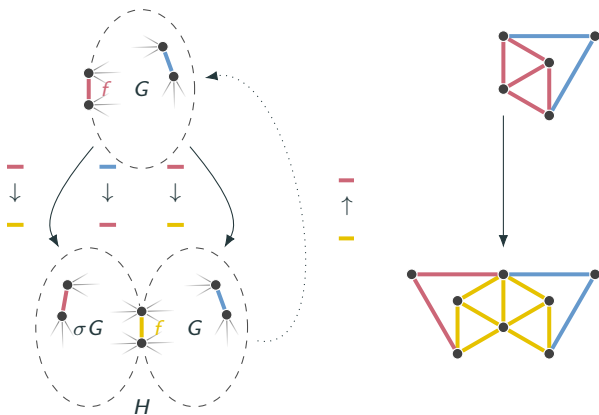
A pseudo-RS-coloring  $\delta$  of graph  $G$  is called an *RS-coloring* if:

1.  $(G, \delta)$  has no almost red-blue cycle, or
2. for every almost red-blue cycle  $C$ , there exist non-gold edges  $e_1, e_2$  in  $C$  and another pseudo-RS-coloring  $\delta'$  of  $G$  (*certificate*) such that  $\delta(e_1) = \delta(e_2)$  and  $\delta'(e_1) \neq \delta'(e_2)$ .



## Theorem

*It is NP-complete to decide whether a reflection-symmetric graph has a pseudo-RS-coloring and it is NP-hard to decide whether a reflection-symmetric graph has an RS-coloring.*



## Necessary conditions

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## **Theorem**

*If  $(G, p)$  is a reflection-symmetric flexible framework, then  $G$  has an RS-coloring.*

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A *valuation* of the complex function field  $\mathbb{C}(\mathcal{M})$  of an irreducible algebraic curve  $\mathcal{M}$  is a map  $\nu : \mathbb{C}(\mathcal{M}) \setminus \{0\} \rightarrow \mathbb{Q}$  such that

1.  $\nu(ab) = \nu(a) + \nu(b)$  for all  $a, b \in \mathbb{C}(\mathcal{M}) \setminus \{0\}$ ,
2.  $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$  for all  $a, b \in \mathbb{C}(\mathcal{M}) \setminus \{0\}$  such that  $a + b \neq 0$ , and
3.  $\nu(\mathbb{C} \setminus \{0\}) = \{0\}$ .

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2 \quad \text{for all } uv \in E_G$$

$$y_{v_0} = 0$$

$$x_{\sigma v} = -x_v, \quad y_{\sigma v} = y_v \quad \text{for all } v \in V_G$$

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$$W_{u,v} = (x_u - x_v) + i(y_u - y_v)$$

$$Z_{u,v} = (x_u - x_v) - i(y_u - y_v)$$

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$$0 = \nu(\lambda(uv)^2) = \nu(W_{u,v}Z_{u,v}) = \nu(W_{u,v}) + \nu(Z_{u,v})$$

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$$\nu(W_{\sigma u, \sigma v}) = \nu(-Z_{u,v}) = \nu(Z_{u,v}) = -\nu(W_{u,v})$$

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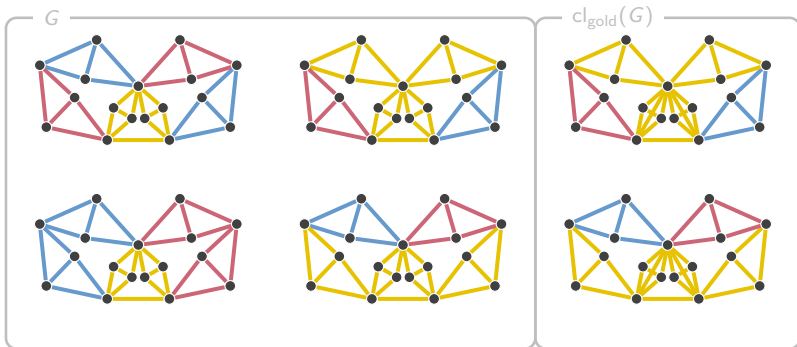
$$\sum_{(u, v) \in C} W_{u, v} = 0 \quad \text{for every cycle } C$$

Define *active* pseudo-RS-coloring  $\delta$  for a suitable  $\alpha \in \mathbb{Q}$ :

$$\delta(uv) := \begin{cases} \text{red,} & \text{if } \nu(W_{u, v}) > \alpha \\ \text{gold,} & \text{if } \alpha \geq \nu(W_{u, v}) \geq -\alpha \\ \text{blue,} & \text{if } \nu(W_{u, v}) < -\alpha \end{cases}$$

G





## Theorem

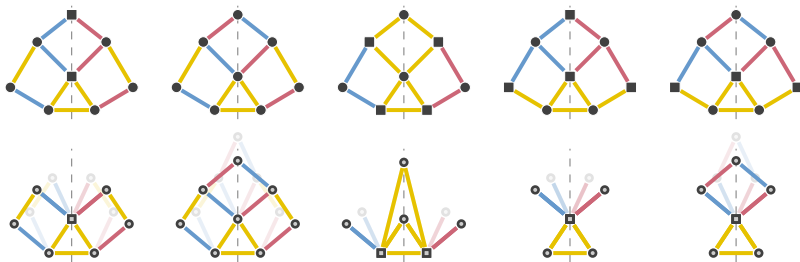
*If  $(G, p)$  is a reflection-symmetric flexible framework, then the gold closure of  $G$  has an RS-coloring.*

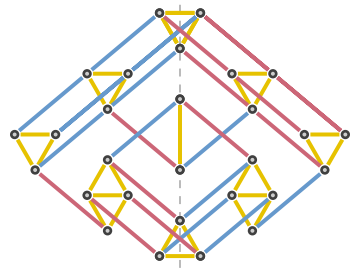
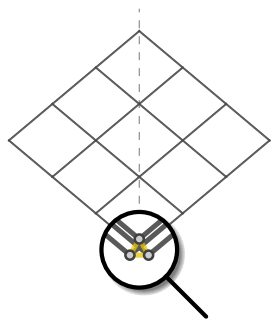
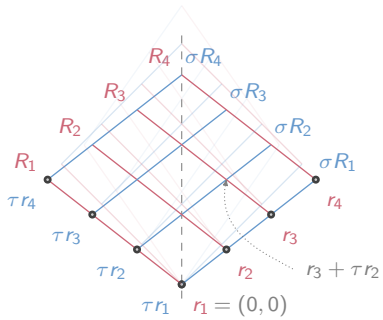
## Sufficient conditions

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## Theorem

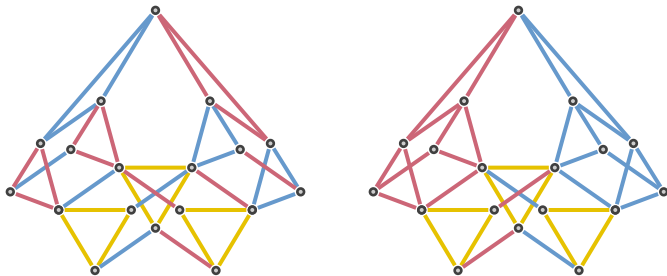
*If  $G$  has an RS-coloring that has no almost red-blue cycle, then there exists a reflection-symmetric flexible framework  $(G, p)$ .*





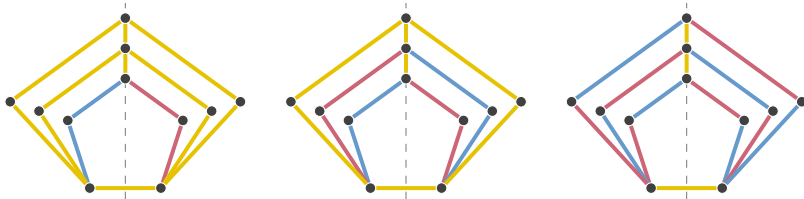
## Theorem

Let  $\delta_1, \delta_2$  be RS-colorings of a graph  $G$  with an invariant vertex such that they are certificates of each other for all almost red-blue cycles and certain 5 technical assumptions hold. Then there is a reflection-symmetric flexible framework  $(G, p)$ .



## Lemma

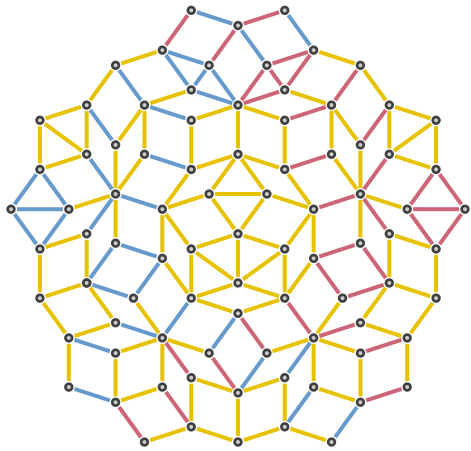
*For every  $\ell \in \mathbb{N}$ , there is a reflection-symmetric graph  $G$  such that every reflection-symmetric motion of  $G$  has at least  $\ell$  active RS-colorings.*



Is there a construction of a reflection symmetric flex from  $\ell$  RS-colorings?

# Frameworks consisting of parallelograms and triangles

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PYRIGI  
pyrigi.github.io

**Thank you**

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